

Geometric Signal Processing on Polygonal Meshes

Gabriel Taubin

IBM T.J.Watson Research Center
<http://www.research.ibm.com/people/t/taubin>

8/24/2000

Taubin / Eurographics 2000 STAR Report

1

Large dense polygonal meshes

- Are becoming standard representation for surface data
 - ◆ 3D Scanning (Reverse engineering, Art)
 - ◆ Isosurfaces (Scientific Visualization, Medical)
 - ◆ Subdivision Surfaces (Modeling, Animation)
- But have too many degrees of freedom (vertices)
- How to ?
 - ◆ Smooth / De-noise
 - ◆ Edit / Deform / Constrain / Animate
 - ◆ Represent / Compress / Transmit
- **BUT FAST !**

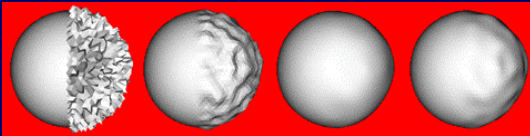
8/24/2000

Taubin / Eurographics 2000 STAR Report

2

Different approaches

- Signal Processing
- Physics-based / PDE Surfaces
- Variational / Regularization
- Multiresolution
- Subdivision Surfaces



8/24/2000

Taubin / Eurographics 2000 STAR Report

3

About this talk

- Initial goal was to present a comprehensive survey
- Final result is not quite comprehensive
- Only way to verify claims is to implement yourself
- Which I did for most algorithms covered in the talk
- But run out of time to implement all
- Demo software (Java) available in my web pages (to be updated soon)
- The talk is biased
- There is much more to understand and do in this area

8/24/2000

Taubin / Eurographics 2000 STAR Report

4

The Signal Processing Approach

- Laplacian smoothing
 - ◆ The shrinkage problem
 - ◆ Fourier analysis on meshes
 - ◆ Smoothing by partial Fourier expansion
 - ◆ Smoothing as low-pass filtering
 - ◆ Taubin $\lambda\mu$ smoothing
 - ◆ FIR/IIR filter design
 - ◆ Implicit Fairing / Multiresolution modeling
 - ◆ Weights / Hard and soft constraints
- Compression of geometry information

8/24/2000

Taubin / Eurographics 2000 STAR Report

5

Main references

- Taubin $\lambda\mu$ smoothing (SG'95)
- Taubin-et-al FIR filter design (ECCV'96)
- Desbrun-et-al Implicit smoothing (SG'99)
- Kobelt-et-al Multiresolution smoothing (SG'98)
- Tani-Gotsman Spectral compression (SG'00)
- Balan-Taubin prediction by filtering (CAD'00)
- Khodakovsky-Schroder-Sweldens Progressive Geometry Compression (SG'00)
- Guskov-et-al Multiresolution Signal Processing (SG'99)
- ...

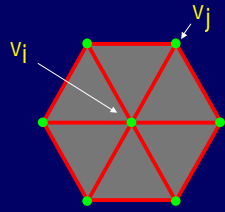
8/24/2000

Taubin / Eurographics 2000 STAR Report

6

Laplacian smoothing in mesh generation

- Used to improve quality of 2D meshes for FE computations
- Move each vertex to the barycenter of its neighbors
- But keep boundary vertices fixed



$$v_i' = \frac{1}{n_i} \sum_{j \in i^*} v_j$$

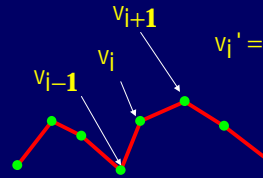
8/24/2000

Taubin / Eurographics 2000 STAR Report

7

Laplacian smoothing of 1D discrete signals

- Known as Gaussian smoothing
- Convolution of 1D signal with Gaussian kernel
- Also for 2D discrete and continuous signals



$$v_i' = \frac{\lambda}{2} v_{i-1} + (1-\lambda) v_i + \frac{\lambda}{2} v_{i+1}$$

$$0 < \lambda < 1$$

8/24/2000

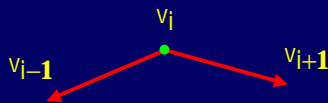
Taubin / Eurographics 2000 STAR Report

8

Laplacian smoothing of 1D discrete signals

$$v_i' = \frac{\lambda}{2} v_{i-1} + (1-\lambda) v_i + \frac{\lambda}{2} v_{i+1}$$

$$v_i' = v_i + \lambda \left(\frac{1}{2} (v_{i-1} - v_i) + \frac{1}{2} (v_{i+1} - v_i) \right)$$



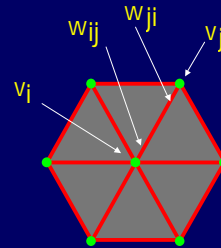
- Preserves DC

8/24/2000

Taubin / Eurographics 2000 STAR Report

9

Laplacian smoothing with general weights



$$v_i' = v_i + \lambda \Delta v_i$$

$$\Delta v_i = \sum_j w_{ij} (v_j - v_i)$$

$$1 = \sum_j w_{ij}$$

$$0 \leq w_{ij}$$

8/24/2000

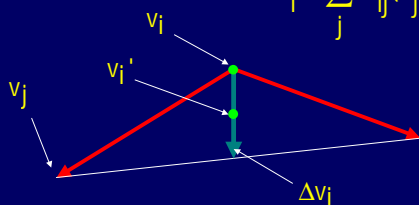
Taubin / Eurographics 2000 STAR Report

10

The Laplacian operator

$$v_i' = v_i + \lambda \Delta v_i$$

$$\Delta v_i = \sum_j w_{ij} (v_j - v_i)$$



8/24/2000

Taubin / Eurographics 2000 STAR Report

11

Laplacian smoothing : advantages

- Linear time
- Linear storage
- Edge length equalization (depending on the application)
- Constraints and special effects by weight control

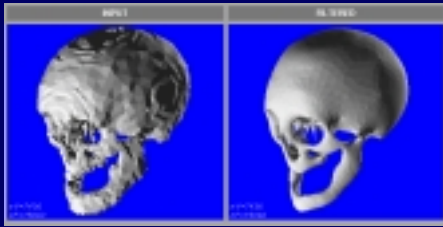
8/24/2000

Taubin / Eurographics 2000 STAR Report

12

Shrinkage of Laplacian smoothing

- DEMO !!!



8/24/2000

Taubin / Eurographics 2000 STAR Report

13

Laplacian smoothing : disadvantages

- Shrinkage
 - ◆ Solve by scale adjustment for closed shapes
 - ◆ What is going on? Stochastic matrices
 - ◆ What is going on? Fourier analysis
 - ◆ Solved by Taubin's algorithm for general shapes
- Edge length equalization (depending on the application)
 - ◆ Fujiwara weights
 - ◆ Desbrun-et-al weights

8/24/2000

Taubin / Eurographics 2000 STAR Report

14

Fixing shrinkage by renormalizing scale



- Adjust scale s to keep distance to barycenter v constant

$$\sum_i \|v_i - v\|^2 = \sum_i \|s(v_i' - v)\|^2$$

$$v_i'' = v + s(v_i' - v)$$

8/24/2000

Taubin / Eurographics 2000 STAR Report

15

Fixing shrinkage by renormalizing scale

- It is a global solution
- Local perturbation changes shape everywhere
- For a better solution we need to understand why shrinkage occurs
 - ◆ Stochastic matrices
 - ◆ Fourier analysis

8/24/2000

Taubin / Eurographics 2000 STAR Report

16

Stochastic matrices

- Square matrices with non-negative elements
- Sum of rows equal to one
- Related to the asymptotic behavior of Markov chains
- Represent probability of transition from state to state

$$M = (m_{ij}) \quad m_{ij} \geq 0 \quad \sum_j m_{ij} = 1$$

- Magnitude of other eigenvalues less than 1
- Powers converge to matrix with eigenvector $\mathbf{1}$ as rows

$$M^n \rightarrow M^\infty$$

8/24/2000

Taubin / Eurographics 2000 STAR Report

17

Stochastic matrix of Laplacian smoothing

$$v_i' = v_i + \lambda \Delta v_i \quad \Delta v_i = \sum_j w_{ij} (v_j - v_i)$$

$$v' = Mv \quad M = (1 - \lambda)I + \lambda W$$

- Converges to the centroid (barycenter) of the vertices

$$v^n = M^n v \rightarrow M^\infty v = \bar{v}$$

- Why? Analyze eigenvalues / eigenvectors

8/24/2000

Taubin / Eurographics 2000 STAR Report

18

Fourier analysis on meshes

$$x_i' = x_i + \lambda \sum_j w_{ij} (x_j - x_i) \quad x' = (I - \lambda K) x$$

- Eigenvalues of $K = I - W$ (FREQUENCIES)

$$0 = k_0 \leq k_1 \leq \dots \leq k_N \leq 2$$

- Right eigenvectors of K (NATURAL VIBRATION MODES)

$$e_0, e_1, \dots, e_N$$

8/24/2000

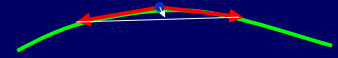
Taubin / Eurographics 2000 STAR Report

19

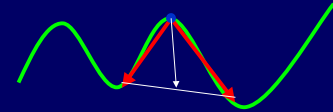
Geometry of low and high frequencies

$$k_h e_{hi} = K e_{hi}' = - \sum_j w_{ij} (e_{hj} - e_{hi})$$

- Low frequency



- High frequency

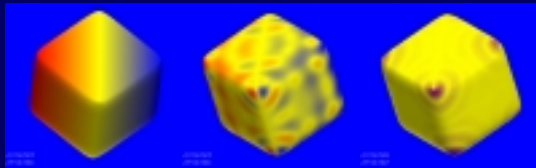


8/24/2000

Taubin / Eurographics 2000 STAR Report

20

Natural vibration modes



8/24/2000

Taubin / Eurographics 2000 STAR Report

21

The Discrete Fourier Transform

- Eigenvectors form a basis of N-space
- Every signal can be written as a linear combination

$$x = \hat{x}_0 e_0 + \hat{x}_1 e_1 + \dots + \hat{x}_N e_N$$

- Discrete Fourier Transform (DFT)

$$\hat{x} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_N)^t$$

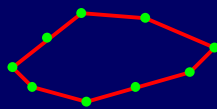
8/24/2000

Taubin / Eurographics 2000 STAR Report

22

The Discrete Fourier Transform

- Corresponds to the classical definition for 1D periodic signals



- For 1D periodic signals there is a fast algorithm to compute the DFT : the FFT
- For the general case of signals defined on irregular meshes, DFT is almost impossible to compute

8/24/2000

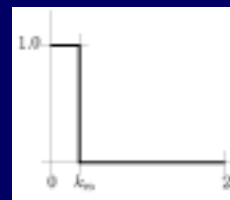
Taubin / Eurographics 2000 STAR Report

23

The Ideal Low-Pass Filter

- Truncated Fourier expansion

$$x' = \hat{x}_0 e_0 + \hat{x}_1 e_1 + \dots + \hat{x}_L e_L$$



$$k_L \leq k_{PB}$$

8/24/2000

Taubin / Eurographics 2000 STAR Report

24

The Discrete Fourier Transform

- Ideal low-pass filtering = truncated Fourier expansion

$$x' = 1\hat{x}_0 e_0 + \dots + 1\hat{x}_L e_L \\ + 0\hat{x}_{L+1} e_{L+1} + \dots + 0\hat{x}_N e_N$$

- But eigenvectors **cannot** be computed !
- Compute an approximation instead : Linear filtering

8/24/2000

Taubin / Eurographics 2000 STAR Report

25

Analysis of Laplacian smoothing

- Laplacian smoothing transfer function

$$x^N = (I - \lambda K) x = f(K) x$$

- f(k) univariate polynomial (rational later)
- f(K) matrix
- K and f(K) have same eigenvectors
- Eigenvalues of f(K)

$$f(k_0), f(k_1), \dots, f(k_N)$$

8/24/2000

Taubin / Eurographics 2000 STAR Report

26

Laplacian Smoothing is a Linear Filter

- After filtering

$$f(K) x = f(k_0) \hat{x}_0 e_0 + \dots + f(k_N) \hat{x}_N e_N$$

- For Laplacian smoothing

$$f(k_0) = 1$$

$$f(k_j) = (1 - \lambda k_j)^N \rightarrow 0 \quad j \neq 0 \quad 0 \leq \lambda < 1$$

- Laplacian smoothing is **not** a low-pass filter !



8/24/2000

Taubin / Eurographics 2000 STAR Report

27

Linear Filtering

- After filtering

$$f(K) x = f(k_0) \hat{x}_0 e_0 + \dots + f(k_N) \hat{x}_N e_N$$

- Evaluation of f(K) x is matrix multiplication
- It **does not** require the computation of eigenvalues and eigenvectors (DFT)

8/24/2000

Taubin / Eurographics 2000 STAR Report

28

Low-Pass Linear Filtering

- After filtering

$$f(K) x = f(k_0) \hat{x}_0 e_0 + \dots + f(k_N) \hat{x}_N e_N$$

- Need to find univariate polynomial f(k) such that

$$f(k_h) \approx 1 \quad k_L \leq k_{PB}$$

$$f(k_h) \approx 0 \quad k_L > k_{PB}$$

- Need to define efficient evaluation algorithm

8/24/2000

Taubin / Eurographics 2000 STAR Report

29

Taubin smoothing (Siggraph'95)

- Two steps of Laplacian smoothing
- First shrinking step with positive factor
- Second unshrinking step with negative factor
- Use inverted parabola as transfer function

$$f(k) = ((1 - \mu k)(1 - \lambda k))^N \quad \text{with} \quad -\mu > \lambda > 0$$



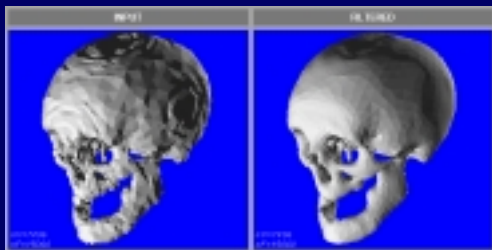
8/24/2000

Taubin / Eurographics 2000 STAR Report

30

Taubin smoothing (Siggraph'95)

- DEMO !!!



8/24/2000

Taubin / Eurographics 2000 STAR Report

31

Taubin-Zhang-Golub (ECCV'96) FIR filter design

- Efficient algorithm to evaluate any polynomial transfer function
- Based on Chebyshev polynomials defined by three term recursion
- All classical Finite Impulse Response (FIR) filter design techniques can be used with no modifications
- Implemented method of "windows" based on truncated Fourier series expansion of ideal transfer function and coefficient weighting to remove Gibbs phenomenon
- DEMO !!!

8/24/2000

Taubin / Eurographics 2000 STAR Report

32

FIR filters vs. IIR filters

- Sharp transitions and narrow pass-bands require very high degree polynomial transfer functions
- Infinite Impulse Response (IIR) filters with rational transfer functions can produce good approximations using polynomials of low degree
- But require the solution of sparse linear systems
- Is it worth the effort ?

8/24/2000

Taubin / Eurographics 2000 STAR Report

33

IIR filters

- If $f(k)=g(k)/h(k)$, with $h(k)$ non-zero in $[0,2]$
- Filtering a signal x requires solving the system

$$h(K) x' = g(K) x$$

- $y = g(K) x$ is an FIR filter
- With $H = h(K)$ solving $H x' = y$ with the Preconditioned Biconjugate Gradients algorithm (PBCG) only requires methods to multiply a vector z by H and by H' and a preconditioner H'

8/24/2000

Taubin / Eurographics 2000 STAR Report

34

Desbrun-Meyer-Schroder-Barr (SG'99) Implicit fairing

- Corresponds to the classical Butterworth filter with transfer function

$$f(k) = \frac{1}{1 + (k/k_{PB})^N}$$

- Need to solve sparse (for small N) linear system

$$(I + (1/k_{PB})^N K^N) x' = x$$

- But with PDE formulation in the paper

8/24/2000

Taubin / Eurographics 2000 STAR Report

35

Implicit fairing

- Laplacian smoothing corresponds to the numerical solution of

$$\frac{\partial x}{\partial t} = \lambda dt \Delta x$$

- using the forward Euler method

$$x' = x + \lambda dt \Delta x = (I + \lambda dt \Delta) x$$

- They use the backward Euler method instead

$$(I - \lambda dt \Delta) x' = x$$

- Stable for large time steps (true or false ?)
- DEMO !!!

8/24/2000

Taubin / Eurographics 2000 STAR Report

36

Kobelt-et-al Multiresolution modeling (Siggraph'98)

- Minimize membrane energy $E_M = \|\Delta x\|^2$
- or thin plate energy $E_{TP} = \|\Delta^2 x\|^2$
- Requires boundary vertex position constraints
- Speed-up by multi-grid approach
- Jacobi updates similar to Laplacian and Taubin updates
- How does it compare with single-res FIR filters ?
- DEMO !!!**

8/24/2000

Taubin / Eurographics 2000 STAR Report

37

Parameters

$$\Delta v_i = \sum_j w_{ij} (v_j - v_i)$$

- Weights
 - Neighborhoods = non-zero weights
 - Prevention of Tangential drift
 - Edge-length equalization
- Boundaries and creases / hierarchical smoothing
- Vertex-dependent smoothing parameters

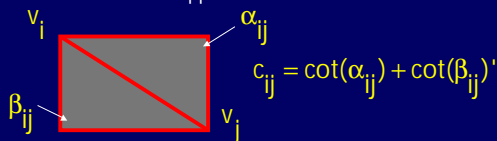
8/24/2000

Taubin / Eurographics 2000 STAR Report

38

Preventing tangential drift

- Fujiwara (P-AMS'95)
 - Weights inversely proportional to edge length
- Desbrun-Meyer-Schroder-Barr (SG'99)
 - Based on better approximation of curvature normal



- Guskov-et-al (SG'99) based on divided differences and second order neighborhood

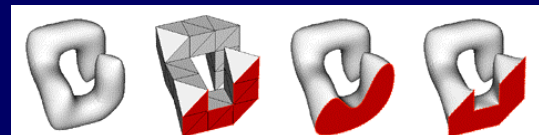
8/24/2000

Taubin / Eurographics 2000 STAR Report

39

Hierarchical neighborhoods

- Assign a numeric label to each vertex
- Vertex j is a neighbor of vertex i only if i and j are connected by an edge, and the label of i is less or equal than the label of j



8/24/2000

Taubin / Eurographics 2000 STAR Report

40

Boundaries and creases

- Use hierarchical neighborhoods
- Assign label 1 to boundary and crease vertices
- Assign label 0 to all internal vertices
- The graph defined by the boundary and crease edges and vertices is smoothed independently of the rest of the mesh
- The rest of the mesh "follows" the graph defined by the boundary and crease edges and vertices

8/24/2000

Taubin / Eurographics 2000 STAR Report

41

Position and normal constraints

- Hard vs. soft constraints
- Hard vertex position constraints are easy to impose
- General hard linear constraints require solving small linear systems
- Yamada-et-al Discrete Spring Model (PCCGA'98) impose soft normal constraints with a spring model that adds an extra term to the smoothing step
- Slow convergence and/or high computational cost
- Multi-resolution helps
- More work needed

8/24/2000

Taubin / Eurographics 2000 STAR Report

42

Geometry compression

- Static or single-resolution vs. progressive
- Connectivity, geometry, and properties
- Geometry and properties cost much more than connectivity
- Commercial grade single-resolution methods available
 - ◆ Taubin-Rossignac Topological Surgery (MPEG-4/ IBM HotMedia)
 - ◆ Touma-Gotsman (Virtue Ltd.)
- Need better geometry prediction/compression schemes

8/24/2000

Taubin / Eurographics 2000 STAR Report

43

Tani-Gotsman (Siggraph'00) Spectral Compression

- Based on partial DFT expansion
- Connectivity is transmitted first
- Encoder computes Eigenvalues/Eigenvectors of matrix K to evaluate Fourier coefficients
- Fourier coefficients are transmitted
- Decoder computes Eigenvalues/Eigenvectors of matrix K to reconstruct the partial sum
- Mesh partition into smaller submeshes to be able to deal with the numerical restrictions
- **Need to compute lots of Eigenvalues/Eigenvectors**

8/24/2000

Taubin / Eurographics 2000 STAR Report

44

Balan-Taubin prediction by filtering (CAD'00)

- Based on a vertex clustering hierarchy (PM, PFS, etc.)
- Connectivity is transmitted progressively interleaved with geometry data
- Fine Geometry is predicted from coarse geometry by filtering the coarse geometry on the fine mesh
- Filter coefficients are determined by solving a LS problem
- Corrections are not transmitted

$$\min_r \|x_f - f(K)x_c\|^2$$

8/24/2000

Taubin / Eurographics 2000 STAR Report

45

Khodakovsky-Schroder-Sweldens Progressive Geometry Compression (SG'00)

- Good for large densely sampled meshes with low topological complexity (3D scanning, etc.)
- MAPS Remeshing produces subdivision surface
- Wavelet compression
- Zero-tree encoding
- Very good results reported

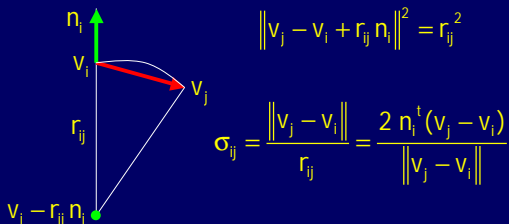
8/24/2000

Taubin / Eurographics 2000 STAR Report

46

Curvature-based Sampling

- Silva-Taubin Curvature-based sampling (SIAM-GD'99)
- Taubin Tensor of curvature (ICCV'95)



8/24/2000

Taubin / Eurographics 2000 STAR Report

47

Conclusion / To Do

- Fast and efficient methods to smooth with hard and soft constraints
- Relation to subdivision surfaces
- Global vs. local behavior of smoothing operators
- Goal: **interactive** free-form modeling based on intuitive user interface to manipulate constraints, remesh, simplify, etc.
- Goal: **practical** and effective methods for the compression of geometry data.
- Implementation of other popular SP operations

8/24/2000

Taubin / Eurographics 2000 STAR Report

48